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Social Networks and Transitive Indifference Graphs

Frank Tutzauer
Department of Communication, University at Buffalo, Buffalo, NY 14260, USA
comfrank@buffalo.edu

A useful way to think of social networks, in particular communication networks, is as a system of actors and the relationships among them. The mathematical theory of graphs, by providing a formal representation of relationships between actors, equips the communication scientist with powerful models of social networks.

This note explores a class of graphs that mimic tightly knit, but diverse factions in communication networks. The graphs involved are rooted in psychological theories of preference and indifference, but the results derived, though in the tradition of Luce (1956) and Roberts (1969, 1970), will be of more interest to communication scientists than to psychologists.

The structure of this note is as follows: First some basic definitions from graph theory are presented. Then, indifference graphs are introduced, and, by requiring transitivity, an equivalence relation is imposed on the actors. A characterization theorem is then proven, followed by a theorem and its corollaries that describe the communication structure of the resulting social networks.

Basic Definitions

By a *relation* R on a set A, we mean a subset of $A \times A$. If $(x,y) \in R$, we write xRy and say x is *R-related* to y (dropping the prefix if the relation is understood). The relation R is *reflexive* if xRx for every $x \in A$, and it is *symmetric* if xRy implies yRx. R is *transitive* if xRy and yRz taken together imply xRz. If R is reflexive, symmetric, and transitive it is called an *equivalence relation*. Let R be an equivalence relation on A, and consider a nonempty subset $B \subseteq A$ along with some element $x \in B$. Then B is called an *equivalence class* if it consists of all the elements of A that are R-related to x.

The ordered pair $G = (V, R)$ is a *graph* if V is a finite, nonempty set, and R is a reflexive, symmetric relation on V. The elements of V are called *vertices* and the elements of R are called *edges*. If xRy, then x and y are said to be *adjacent*. The *neighborhood* of a vertex is the set of all vertices to which it is adjacent. We say the graph $H = (W, S)$ is a *subgraph* of $G = (V, R)$, and we write $H \subseteq G$, if $W \subseteq V$ and $S \subseteq R$. If S is a proper subset of R, or if W is a proper subset of V, we then we say that H is a *proper subgraph* of G. It is an *induced subgraph* if whenever two vertices of W are R-related they are also S-related.

Two vertices are *structurally equivalent* if their neighborhoods are identical (cf. Lorrain & White, 1971). Because structurally equivalent vertices cannot be distinguished from each other on the basis of their links to other vertices (because structurally equivalent vertices are adjacent to exactly the same vertices), it is sometimes convenient to "collapse" them into a single vertex. More formally, given a graph $G = (V, R)$, the *reduced* graph of G, denoted G^*, is the graph that results when one takes the (structural) equivalence classes of V as the vertices of G^* and one considers two equivalence classes to be adjacent in G^* if and only if their members are adjacent in G.

A *path* (of length n) from vertex x_0 to vertex x_n is an ordered sequence of distinct vertices $P = \{x_0, x_1, . . . , x_n\}$ with each vertex in the sequence being adjacent to the vertex following it. P is a *geodesic* if it is a shortest such path, i.e. if there is no other path from x_0 to x_n of length less than n. The *distance* between a pair of vertices is taken to be the length of the geodesic between them. The graph $G = (V, R)$ is *connected* if for every pair of vertices x and y there is a path from x to y. The *diameter* of a connected graph is the length of its longest geodesic, and a graph of unit diameter is called *complete*. A connected subgraph of G is a *component* if it is maximal connected, i.e. if there is no other subgraph $J \subseteq G$ properly containing H. The vertices in a component are all connected by paths, whereas there are no paths whatsoever between vertices of different components. Components thus represent closed subsystems.

Suppose for a subgraph $H \subseteq G$ we discover that every pair of vertices in H are separated by a distance (in G) of no more than n. If H is a maximal such subgraph it is called an *n-clique* (cf. Luce, 1950). A connected n-clique of diameter n is called, following Alba (1973), a *sociometric clique* (of diameter n). Suppose H and J are two sociometric cliques with the same diameter belonging to G. It is possible some vertices belong to more than one sociometric clique, and thus H and J may share members. If they do not, i.e. if their vertex sets have empty intersection, then H and J are *separated*. A vertex of H adjacent to a vertex of J, but not belonging to J is called a *bridge*, whereas vertex of G, adjacent to at least one vertex of H and at least one vertex of J, but belonging to neither, is called a *liaison*. Finally, a vertex x of the graph $G = (V, R)$ is an *extreme point* if xRy and xRz together imply yRz and the existence of some distinct point $w \in V$ such that wRy and wRz, but not wRx.

The Structure of Transitive Indifference Graphs

Following Roberts (1969, 1970), we have:

Definition. A graph $G = (V, R)$ is an *indifference graph* if, for every connected, induced subgraph $H \subseteq G$, the reduced graph H^* has only one vertex, or exactly two extreme points. □

Recall that the vertices of H^* are equivalence classes. Thus if there is but a single vertex in H^* then the vertices of H are all structurally equivalent to each other. Alternatively, if there are exactly two extreme points in H^*, then the vertices of H can be ordered in the relation R so that there is a "largest" vertex and a "smallest" vertex.

Indifference graphs generate interest from a psychological viewpoint because V can be thought of as a set of alternatives on which the relation R specifies indifference. To the communication theorist, indifference graphs represent relationships between the members of certain types of social systems. For example, if we measure some attribute on individuals and postulate that the individuals will form communication ties if their attributes are sufficiently similar, the resulting mathematical representation will be an indifference graph. One can see this by means of a result deduced by Roberts (1969), namely that one can define a real-valued function on the vertices such that two vertices are adjacent if and only if their function values are within some arbitrary positive constant (typically scaled to be 1, for convenience).

Among the first questions one might ask of a communication network is whether or not it is transitive. For example, Monge and Contractor (2003) find transitivity in the communication patterns of 17 individuals negotiating a research and development agreement, and Murshed, Uddin, and Hossain (2015) found that transitivity in a communication network increased as a result of organizational crisis. Indeed, a fundamental tenet of Granovettor's strength-of-weak-ties perspective is that the stronger the tie between a pair of individuals, the greater the overlap in their personal networks (neighborhoods). As Wigand (1977) has claimed, transitivity is one of the "major properties of relational constraints . . . " (p. 182). Transitivity in indifference graphs is thus a natural concept to investigate. The following theorem characterizes the structure of such networks.

Theorem 1 (Characterization). The following statements are equivalent for the graph $G = (V, R)$:

- *(i)* G is a transitive indifference graph.
- *(ii)* R is an equivalence relation on V.
- *(iii)* All components of G are complete.
- *(iv)* For any connected, induced subgraph $H \subseteq G$, the reduced graph H^* consists of exactly one vertex.
- *(v)* There exists a function f mapping vertices to the real numbers such that for any vertices x and y, we have xRy if and only if $f(x) = f(y)$.

Proof. We will prove the cycle of implications *(i)* → *(ii)* → . . . *(v)* → *(i)*. To see *(i)* → *(ii)*, note that because G is a graph, R is reflexive and symmetric, and by assumption it is transitive. Therefore R is an equivalence relation. To prove *(ii)* → *(iii)*, suppose that H *is a component of* G and let x and y be arbitrary vertices of H. H a component implies it is connected, which in turn implies there is a path from x to y. Because R is an equivalence relation, we must have xRy, which means the geodesic between them is of unit length. Because x and y were chosen arbitrarily, H must be of unit diameter, and therefore complete. For *(iii)* → *(iv)*, let H be an induced, connected subgraph of G. Because H is connected it is a subgraph of some component $J \subseteq G$. Because J is complete, and because H is induced, H must be complete as well. Therefore all of its vertices have the same neighborhoods and thus every vertex is structurally equivalent to every other vertex, which means there is but a single equivalence class for H^*. For *(iv)* → *(v)*, denoted by $H_1, H_2, \ldots, H_n$ the n components of G. Let $f(x) = i$ for every vertex of H_i, $i = 1, 2, \ldots, n$. Clearly xRy implies $f(x) = f(y)$. Furthermore, if $f(x) = f(y)$ then x and y come from the same component. And since for all i, H_i^* consists of a single equivalence class, x and y must be structurally equivalent, and therefore adjacent in G. Thus xRy if and only if $f(x) = f(y)$. Finally, we prove *(v)* → *(i)*. Clearly $f(x) = f(y)$ implies the difference of these two values is 0 which, by Roberts' (1969) result, means that G is an indifference graph. Furthermore, R is transitive since, for any x, y, and z such that xRy and yRz, we have $f(x) = f(y)$ and $f(y) = f(z)$, implying $f(z) = f(z)$ and therefore xRz. □

Theorem 1 provides a representation for transitive indifference graphs and can be used to explore clique structure. These highly cohesive subgroups exhibit relatively few communication links with the rest of the system, but high internal connectedness. In the extreme case, "we should expect communication among group members to increase within subgroups, but to decrease between members of different subgroups. The total 'group' might disintegrate into several warring factions" (Collins & Raven, 1969, p. 125). As it turns out, it is precisely such highly factional systems that correspond to the various transitive indifference graphs. The correspondence is explored in the following theorem and its corollaries.

Theorem 2 (Clique Structure). Let $G = (V, R)$ be a transitive indifference graph. Every n-clique contained in G is a 1-clique.
Proof. Suppose not, i.e. suppose there exists an n-clique of G having two of its points, x_0 and x_n, joined by a geodesic in G of length 2 or more. Suppose that this geodesic has vertex set $\{x_0, x_1, \ldots, x_n\}$ and let H be the subgraph of G induced by this set. Because G is a transitive indifference graph, and because H is connected and induced, the reduce graph H^* consists of but a single equivalence class, implying x_0 is adjacent to x_n. Thus the geodesic joining these two vertices is of unit length, contradicting the assumption that it is of length 2 or more, and establishing the theorem. □

Corollary 1 (Connectedness). All n-cliques of G are connected.
Proof. In any graph, the 1-cliques are complete, and therefore connected. □

Corollary 2 (Tightness). Every component of G is a sociometric clique of diameter 1.
Proof. Every component is an n-clique, and thus a 1-clique. A 1-clique, being complete, has unit diameter, and is, by definition, a sociometric clique of diameter 1. □

Corollary 3 (Separation). All distinct sociometric cliques of G are separated.
Proof. Suppose not. Then there exist distinct sociometric cliques $H = (W, S)$ and $J = (X, T)$ such that the intersection of W and X is nonempty. It is easy to show that if x is in the intersection, it is adjacent to all elements of W and X. This implies that the subgraph induced by the union of W and X is connected and consequently contained in a 1-clique. Therefore, because H and J are distinct, at least one of them must be properly contained in a 1-clique, which is the contradiction establishing the corollary. □

Corollary 4 (Remoteness). G is devoid of bridges and liaisons.
Proof. Let $H = (W, S)$ and $J = (X, T)$ be sociometric cliques. By Theorem 2, they are 1-cliques and therefore complete. Choose x arbitrarily and suppose it to be adjacent to some $y \in X$. The transitivity of R and the completeness of J imply that x is adjacent to all vertices of X, so therefore x cannot be a liaison. Moreover, it cannot be a bridge because if x were adjacent to some member of W, transitivity would imply that all vertices of X were adjacent to all vertices of W, contradicting that fact that J is maximal. Thus, x is not a bridge. □

Corollaries 3 and 4, taken together, imply that each clique in a transitive indifference graph is isolated from the rest of the system, while Corollaries 1 and 2 speak to the internal cohesiveness of such subgroups. Transitive indifference graphs, therefore, capture the essence of a system marked by “warring factions.”

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