An Iterative Solution for Eigenvalues and Eigenvectors*

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Let A be a given matrix of order $n * n$, and let b_{11} be an arbitrary $n * 1$ vector. Calculate $Ab_{11} = b_{12}$; b_{12} is also an $n * 1$ vector. Calculate $Ab_{12} = b_{13}$. If we continue this process, it will ultimately lead to a vector b_{1p} that is equal to the first eigenvector of A, so that Ab_{1p} is proportional to b_{1p}.

A good practical procedure is to start with a trial vector b_{11} in which all elements are equal to one. Then b_{12} will give row totals of A. Then divide b_{12} by its largest element, so that in the resulting vector the largest element is equal to one. Continue in this way until $b_{1,p+1}$ is proportional to b_{1p}; i.e., if we divide Ab_{1p} by its largest element, we get b_{1p} back; the divider is then equal to the eigenvalue.

The method can be shown to converge to the eigenvector that has the largest eigenvalue associated with it. For an illustration, we take the matrix in table 12.1(Table 1). The first trial vector is b_{11} with unit elements. The output vector is

$$Ab_{11} = \begin{pmatrix} 2.623 \\ 2.776 \\ 3.025 \\ 2.828 \\ 2.864 \end{pmatrix},$$

which, divided by 3.025, becomes

$$b_{12} = \begin{pmatrix} .8671 \\ .9176 \\ 1.0000 \\ .9348 \\ .9467 \end{pmatrix}.$$

Then calculate

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Table 1: Correlations between five variables related to social stratification.

	x_1	x_2	x_3	x_4	x_5
x_1	1.000	.516	.453	.332	.322
x_2		1.000	.438	.417	.405
x_3			1.000	.538	.596
x_4				1.000	.541
x_5					1.000

$$Ab_{12} = \begin{pmatrix} 2.4088 \\ 2.5764 \\ 2.8619 \\ 2.6556 \\ 2.6994 \end{pmatrix},$$

and divide by 2.8619 to obtain

$$b_{13} = \begin{pmatrix} .8416 \\ .9002 \\ 1.0000 \\ .9278 \\ .9431 \end{pmatrix}.$$

Continuing in this way, we find ultimately

$$Ab_{18} = A \begin{pmatrix} .8336 \\ .8942 \\ 1.0000 \\ .9275 \\ .9439 \end{pmatrix} = \begin{pmatrix} 2.3599 \\ 2.5314 \\ 2.8309 \\ 2.6259 \\ 2.6723 \end{pmatrix};$$

dividing by 2.8309,

$$b_{19} = \begin{pmatrix} .8336 \\ .8942 \\ 1.0000 \\ .9275 \\ .9439 \end{pmatrix}.$$

We see that $b_{19} = b_{18}$, and we have the eigenvector accurate to four decimal points, with eigenvalue 2.8309. If b_{19} is normalized to the eigenvalue 2.8309, we have the first column of table 13.6(Table 2).

In order to find subsequent eigenvectors, normalize the eigenvector we have found to the eigenvalue. Let us call the result f_1. Then calculate the residual matrix $A_1 = A - f_1 f_1'$. In the example, the result is

Table 2: Factor matrix for correlation matrix given in table 12.1.

	f_1	f_2	f_3	f_4	f_5
x_1	.6806	.5896	-.3095	.2122	.2192
x_2	.7301	.4152	.4520	-.2703	-.1301
x_3	.8165	-.1758	-.3366	-.0394	-.4329
x_4	.7573	-.3378	.3022	.4677	.0453
x_5	.7707	-.3958	-.0952	-.3491	.3437
eigenvalue	2.8310	.8219	.5140	.4604	.3727

$$A_1 = \begin{pmatrix} .5366 & .0189 & -.1028 & -.1835 & -.2026 \\ .0189 & .4668 & -.1581 & -.1360 & -.1578 \\ -.1028 & -.1581 & .3332 & -.0804 & -.0333 \\ -.1835 & -.1360 & -.0804 & .4263 & -.0427 \\ -.2026 & -.1578 & -.0333 & -.0427 & .4058 \end{pmatrix}.$$

Then the procedure is repeated for this residual matrix. The result would be

$$A_1 b_{21} = \begin{pmatrix} .0666 \\ .0338 \\ -.0415 \\ -.0164 \\ -.0307 \end{pmatrix};$$

dividing by .0666.

$$b_{22} = \begin{pmatrix} 1.0000 \\ .5079 \\ -.6233 \\ -.2468 \\ -.4619 \end{pmatrix}.$$

Continuing, we would find at the 21st trial that

$$b_{2,21} = \begin{pmatrix} 1.0000 \\ .7042 \\ -.2982 \\ -.5728 \\ -.6713 \end{pmatrix},$$

and that this vector is proportional to $b_{2,22}$, with proportionality factor equal to .8219, the second eigenvalue.

The procedure in this second stage could be speeded up considerably if we first change the sign of any row and column in A_1 where there are many negative elements. In the example, this might be done for the last three rows and columns, so that the only negative elements remaining would be a_{34}, a_{35}, and a_{45}, and their symmetric counterparts, a_{43}, a_{53}, and a_{54}. The procedure then would converge much more rapidly. We should keep in mind, however, that for the result of the procedure we must change signs again for the three last elements of the vector.

A second residual matrix A_2 is then obtained from $A_1 - f_2 f_2'$, and the procedure can be repeated until the final eigenvector is determined.

Another way to speed up computations is to apply the procedure to a power of A, such as A^2 or A^3. This does not change eigenvectors, but it results in proportionality factors that are also powers (of the same degree as for A) of the eigenvalues of A.

We might also warn that enough decimal places must be carried, since otherwise residual matrices can become increasingly in error and later eigenvectors will be inaccurate.