ROTATION TO CONGRUENCE FOR GENERAL RIEMANN SURFACES UNDER THEORETICAL CONSTRAINTS

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ABSTRACT

Computational procedures which calculate eigenvectors as intermediate or final results of data analysis (such as Factor Analysis and multidimensional scaling), are confounded in the comparison of time-series or multiple sample data sets by the arbitrary orientation of the eigenvectors across data sets. While the elimination of these arbitrary differences in orientation is known to be possible by sequences of rotations and translations, special difficulties are encountered when the hyperspaces spanned by the eigenvectors are generally Riemannina rather than Euclidean. The present article discusses these difficulties and presents a general method for comparing time-series or multiple-sample data sets of a Riemannian type under general theoretical constraints.

Worked through examples are presented.

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Introduction

By now it is commonplace to understand that the notion of absolute motion and absolute change has no meaning, but rather that any motion must be gauged relative to some arbitrary reference frame. In physical work, arbitrary but conventional reference frames are often suggested by the character of the situation under study, and particularly for terrestrial motions, the surface of the earth frequently serves as a suitable choice. Thus for most practical terrestrial motions, the surface of the earth may be regarded as fixed, and motions of other objects may be calculated relative to the earth's surface.

In most examples of cognitive or cultural change, however, conventional reference frames are seldom obvious. Thus, for example, an individual who is regarded as at one time conservative by his or her reference group may be regarded at another time as radical by the same group. The individual, however, may regard his or her position as unchanged, but view the reference group as increasingly conservative. The absence of a standard reference frame for social and cultural research has made comparisons of research findings across observers and times problematic, and without doubt has complicated the development of kinematic and dynamic theory within the human disciplines to a great degree.

Among quantitative social scientists, the question of reference frames has been dealt with most precisely by psychometricians and communication researchers within the area of multidimensional scaling (MDS). Within MDS, measurements made on arbitrary scales are reexpressed on (generally orthogonal) coordinate reference axes which serve as a frame of reference within which the objects measured may be arrayed. When measurements have been made at multiple times or on multiple samples, however, the orientation of the reference axes in each space are generally arbitrary with regard to each of the others. This is equivalent to the well-known mechanical problem of comparison of events and processes across reference frames which are in relative motion with regard to each other.

Within psychometrics, many solutions to this carefully studied problem have been proposed, all of which include at some stage rotations and usually translations, while some allow as well for change of scale (central dilation). (N. Cliff, 1966; Schönemann, 1966; Lissitz, Schonemann, Lingoes, 1978).

In spite of the care with which these areas have been scrutinized, not all areas of concern have been discussed explicitly in the literature. Two issues in particular form the focus of this discussion. First, this article will discuss the question of establishing theoretical constraints on the general solution to the rotation problem, such, for example, as taking some subset of measured objects as a frame of reference rather than the entire set. While this problem has been dealt with elsewhere (Lissitz, et al., 1978) a general solution for other than Euclidean real spaces is not available. Thus, the second focus of the present paper is a generalization of the rotation problem from real cartesian coordinates to generalized Riemannian spaces.

Theoretical constraints

The typical "procrustes" rotation problem requires finding a (generally orthogonal) transformation which minimizes some "difference" function between two data matrices. (Cliff, 1966; Schoemann, 1966). Since the transformation desired is generally one which leaves the dissimilarity relations within each dataset invariant, the transformation matrix T usually consists of a set of pairwise rotations of axes. While this is well-known, it is not often made explicit that such transformations are only commutative when the rotations are infinitesimal (Goldstein, 1951). Since truly infinitesimal rotations are not possible, in practice it is necessary to perform a succession of iterations with a small finite angle of rotation. Thus, such a routine would adjust all possible pairs of coordinate axes by a small amount, check the value of the difference function between the (now adjusted) data matrices, then repeat the operation again through all pairs of axes, check again, and so on until the difference function can no longer be reduced. (Attempts to minimize the difference function for each pair of axes in succession will not in general achieve a global minimum.) In the algorithm to be described here, a given pair of axes is rotated 1 degree, the difference function is evaluated and compared to the starting value of the difference function. If this new value is higher than the old value, the original matrix of data is restored and the same pair of axes is rotated one degree in the opposite direction. If this results in a reduction of the difference function the operation is not repeated, but rather a second pair of axes is selected and the operation is performed for this second pair. Only after all pairs of axes have been adjusted in this way does the routine pass through the set of pairs of axes again.

The most common difference function (and the one used in the current algorithm, with some modifications we will discuss below) is given by the squared distances among corresponding datapoints in two multidimensional configurations summed over all the points, or

$$s^2 = g_{\mu\nu} R^{\mu}_{(\alpha)} \tilde{R}^{\nu}_{(\alpha)} = \min \quad . \tag{1}$$

$$\mu = 1, 2, \ldots, r$$
$$\nu = 1, 2, \ldots, r$$
$$\alpha = 1, 2, \ldots, k$$

In expression (1), the matrix $g_{\mu\nu}$ has the familiar form

$$g = 1 \text{ if } \mu=\nu$$
$$= 0 \text{ if } \mu\neq\nu \qquad (2)$$

and we are following the Einstein convention that all repeated indeces are to be summed over. The $R^{\mu}_{(\alpha)}$ and $\tilde{R}^{\mu}_{(\alpha)}$ refer to the projections of the datapoints on the two sets of orthonormal reference axes, with the superscript designating the axes and the subscript being the label of the datapoint. Lower case r is the number of axes, and k is the number of datapoints projected on the coordinates.

When the $g_{\mu\nu}$ have the form given in (2), expression (1) reduces to the ordinary Euclidean distance function defined for orthonormal coordinate axes and little advantage is to be gained from this notation over a more conventional form. If we allow the $g_{\mu\nu}$ to represent the scalar products of the coordinate axes, however, then the entries on the principle diagonal will represent the squared lengths of the coordinate axes and the off-diagonal axes will be given by

$$g_{\mu\nu} = \tfrac{1}{2} |e_{\mu}| \; |e_{\nu}| \cos\alpha_{\mu\nu} \quad ,$$

where e_{μ} and e_{ν} represent the basis vectors of the configuration, and where $\alpha_{\mu\nu}$ represent the angle between the μth and νth axes. Thus (1) becomes the general distance function for non-orthonormal coordinate axes (Einstein, 1951; McConnell, 1933). (If the coordinate axes are curvilinear, expression (1) must be replaced by the differential form[1]

$$ds^2 = g_{\mu\nu} dR^{\mu}_{(\alpha)} d\hat{R}^{\nu}_{(\alpha)} \qquad (3)$$

but such an analysis is beyond the scope of the present paper.)

Successive applications of the transformation matrix T through all sets of pairs of axes until (1) is at a minimum will in general serve to match arbitrarily oriented datasets and thus will serve as a convenient frame of reference against which changes in the configuration of datapoints can be calibrated as long as no prior empirical or theoretical knowledge of these changes is available. In many interesting cases, however, such knowledge is available, and so additional constraints ought to be applied to the solution. More specifically, often an investigator may have reason to suspect that some of the datapoints have exhibited little or no change across the interval of measurement, or, alternatively, should for any reason be expected to be the same across datasets, while other datapoints ought to be expected to have changed their locations. Such might be the case in a laboratory experiment, for example, in which some datapoints have been manipulated while others have been controlled. In such a case, the solution ought to be constrained such that the difference function should be minimized only for those datapoints expected to be stable. Lissitz, et al. (1978) have provided a solution of this problem for real, Euclidean datasets, but have not generalized their solution for any Riemann space.

Two steps are required for this operation. First, the coordinate system must be translated such that the center of both datasets lies at the center of the subset of stable datapoints. If we designate the subset of stable datapoints as $\hat{R}^{\mu}_{(\beta)}$ and the number of such stable datapoints as $\hat{k}$, then the center is given by the vector

$$\sum_{\beta}^{\hat{k}} \hat{R}^{\mu}_{(\beta)}/\hat{k} \tag{4}$$

and the desired translation by

$$\overline{R}^{\mu}_{(\alpha)} = R^{\mu}_{(\alpha)} - \sum_{\beta}^{\hat{k}} \hat{R}^{\mu}_{(\beta)}/\hat{k} \tag{5}$$

where $\overline{R}^{\mu}_{(\alpha)}$ = the translated coordinates of the αth vector.

Once both datasets have been translated to a common origin, either may be rotated toward the other by means of the iterative application of the matrix of small finite pairwise rotations, but the difference function which is minimized must be modifed so that the distances among the datapoints expected to change position are not included in the quantity to be minimized. All datapoints are rotated, of course, but the distance from each "free" or unconstrained datapoint to its counterpart in the second dataset is not added into the distance function (1).

Tables one through five illustrate these procedures for an arbitrary real three-dimensional configuration of five datapoints. As table one shows, a three dimensional configuration of five datapoints was constructed, then the distance relations between the third datapoint and all others were arbitrarily modified. Given the arbitrary nature of these data, it is certain that a reference frame can be identified (consisting of the 1st, 2nd, 4th and 5th datapoints) relative to which only the third point will exhibit motion. Table two shows the eigenvectors of these configurations, which were obtained from a standard diagonalization of the centroid scalar products of the distance matrices (Torgerson, 1958) via the Galileo™ version 4.5 computer program at the East-West Center in Honolulu. Table three gives the eigenvectors translated to the centroid of points one, two, four and five, and table four gives the rotated coordinates, as well as the distances of each of the datapoints from its counterparts in the second dataset. Table five gives relevant data about the orientations of the position vectors of the datapoints in the now common coordinate reference frame. As these data make clear, only the third datapoint exhibits any motion. (The figures show minor error due to the one degree increment of the rotation, such that the solution can be accurate to only $\pm .5^{\circ}$, but the solution can be made arbitrarily accurate at somewhat greater expense by reducing the magnitude of the rotation angle.)

Rotation in Generalized Riemann Space

As noted, the data in tables one through five are real and Euclidean. Most psychometricians restrict themselves to such matrices for a variety of reasons, but there is increasing evidence against this practice. First, arguments that spaces representing psychological or cultural processes ought to be real, Euclidean and of small dimensionality have never been particularly forceful, and in fact much psychological theory is inconsistent with such a view. In particular, balance theories and dissonance theory clearly suggest that human conceptions are particularly prone to inconsistency and illogicality (Heider, 1957; Festinger, 1957; Newcomb, 1951). Elaborate models have been developed to deal particularly with what people do when they discover their own inconsistencies. There is good reason, furthermore, to believe that violations of triangle inequality relations in pair comparison magnitude estimation tasks provide a useful measure of such inconsistencies (Woelfel, Barnett and Dinkelacker, 1977). Communication researchers in particular have observed regular and statistically reliable violations of triangle inequality relations for cultures and segments of cultures (Barnett and Woelfel, 1979; Woelfel, et al., 1978; Woelfel and Fink, forthcoming). While it is beyond the scope of this paper to argue the theoretical or empirical merits of such a view, it should suffice to point out that there exists a large and growing body of workers in several fields who make increasing use of metric scaling analyses of highly reliable data which violate triangle inequality relations. These violations result in the presence of one or more imaginary eigenvectors in the solutions of multidimensional scaling problems. In general, any space whose distance function is given by (1) above is a Riemann space (McConnell, 1933, p.246). When the $g_{\mu\nu}$ (usually called the metric or fundamental tensor) takes on the values given in (2), the space is Euclidean. We have already considered the generalization to non-orthonormal coördinates earlier (p.), and an extension to general curvilinear coordinates in (3). It is easy to show that the presence of imaginary eigenvectors in the solution requires that the corresponding eigenvalues be negative real numbers, and this in turn can be accounted for fully by allowing some of the diagonal values of the metric tensor to become negative. If we eliminate the restriction that the matrix of the $g_{\mu\nu}$ be positive, therefore, we have a general method for defining a distance function for any Riemann space, in distinction from some of the more specialized and restricted recent treatments (see, for example, Piesko, 1976; Lindman & Caelli, 1978).

Since the process of diagonalization common to most multidimensional scaling programs makes it possible without exception to choose orthogonal reference axes, and since it is similarly always possible to normalize the solution such that the basis vectors of the space are unit vectors with no loss in information, we suffer no important losses of generality if we define the metric tensor as

$$g_{\mu\nu} = \begin{array}{l} 0 \text{ if } \mu \neq \nu \\ 1 \text{ if } \mu = \nu < p \\ -1 \text{ if } \mu = \nu \geq p \end{array} \qquad (6)$$

$$\mu, \nu = 1, 2, \ldots, r$$

where the p_{th} through r_{th} eigenvalues are negative, allowing the p_{th} through r_{th} eigenvectors to take on imaginary values.

Subject to these generalizations, the form (1) remains a useful difference function to be minimized in the rotation algorithm.

Since the choice of pairwise rotations was itself dictated in part by a desire to find a transformation which leaves the distance relations within each dataset invariant, however, we may no longer apply the transformation matrix T sequentially through all possible pairs of reference axes. This is due to the fact that a mixed rotation, that is, the rotation of a real and imaginary axes through any angle does not in general leave distances within the complex plane invariant. (The reader can verify this quickly by considering any rotation of the vector $x=(1,i)$ through an arbitrary angle. Since the length is $1^2 + i^2 = 0$, general rotations obviously leave the length in the rotated coordinate system non-zero).

This problem is easily solved, however, when we recall that any complex function may be separated into its real and imaginary part, (Cushing, 1975) so that we may partition the datasets into their real parts and their imaginary parts, carry out the pairwise rotations separately within each part, then rejoin the parts after (1) has been minimized. (Since the submatrices of each dataset need not in general be conformable across all datasets, it will usually be necessary to augment the sets of lower rank by adding vectors of zeros, but this operation does not affect the outcome in any way, even though it is fairly tedious to accomplish in FORTRAN.)

Table six shows a set of arbitrary distances among five points which violate triangle inequality relations on a small scale. Tables seven and eight represent respectively the eigenvectors ("normal coordinates" in the Galileo version 5.2 output[2]) and rotated coordinates representing this configuration. These violations are small, but sufficient to produce small imaginary loadings in the first dataset and somewhat larger imaginary loadings in the second. (The absolute size of the imaginary loadings is of no significance, as long as their magnitude relative to the real loadings is not sufficient to make the overall length of any vector imaginary, which confuses the rotating algorithm in this particular program). Table nine shows the overall goodness of fit of the rotated matrix to the target, and once again the algorithm "correctly" attributes motion only to the third datapoint. (As remarked earlier, the finite 1° angle of rotation of the Galileo version 5.2 program is responsible for minor departures from the correct solution.)

By way of contrast, table ten gives the coordinates of the same dataset rotated to an ordinary least squares best fit, that is, without allowing the third datapoint to be free. Table eleven shows goodness-of-fit data for this rotation, and, although concept three still exhibits the most motion, all the other concepts exhibit a non-trivial motion as well. Figure 1 shows the first principle plane of the configuration with and without the fcons option to give some visual impression of how much difference the algorithm makes in practice.

Conclusions

Rational determination of relative motion and change requires stipulation of some reference frame with regard to which such changes may be calibrated. In multidimensional scaling studies, this problem requires establishing an invariant set of coordinates against which change processes may be arrayed. When no information about the change process is available a priori, ordinary least squares "procrustes" rotations provide a best attempt at such a solution. When information about the stability and change of the datapoints can be provided in advance, however, the ordinary least squares algorithm is no longer optimal, but rather a weighted solution is required. In this article, the simplest such weighting is discussed, that is, one in which datapoints thought to be stable are included in the minimization function while those expected to move are left out. The solution is generalized to include any Riemann space subject only to the constraint that only datapoints whose lengths are real are included.

The specific algorithm used in the illustration, (Galileo version 4.5 and Galileo version 5.2) is an iterative pairwise rotation scheme which, while analytically acceptable, is undoubtedly slower than more recent and more sophisticated eigenvector routines such as that provided by Lissitz, et al. (1978). While these authors have not attempted to do so, modification of these more advanced algorithms for general Riemann spaces ought to prove straightforward. Specifically, partitioning the eigenvectors into the set of real and the set of imaginary eigenvectors, augmenting as needed and applying the Lissitz, et al. procedure within each set ought to produce the result shown here to higher levels of precision at some computational savings.

TABLE ONE: DISTANCES AMONG FIVE POINTS IN A THREE DIMENSIONAL CONFIGURATION AT TWO POINTS IN TIME

SET ONE

DATAPOINT	1	2	3	4
1	0.000	0.000	0.000	0.000
2	8.000	0.000	0.000	0.000
3	6.708	7.810	0.000	0.000
4	5.385	6.708	5.657	0.000
5	10.488	7.874	6.403	10.630

SET TWO

DATAPOINT	1	2	3	4
1	0.000	0.000	0.000	0.000
2	8.000	0.000	0.000	0.000
3	3.606	5.385	0.000	0.000
4	5.385	6.708	4.000	0.000
5	10.488	7.874	7.550	10.630

TABLE TWO: EIGENVECTORS AND EIGENVALUES OF THE CONFIGURATIONS PRESENTED IN TABLE ONE

SET ONE

DATAPOINT	COORDINATES				
	1	2	3	4	5
1	−3.882	−1.207	2.998	0.000	0.006
2	0.645	4.680	0.023	0.012	−0.001
3	0.758	−2.977	−1.511	0.021	−0.001
4	−3.855	0.226	−2.192	−0.019	0.005
5	6.335	−0.722	0.682	−0.015	−0.009
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--					
	71.057	32.794	16.545	0.001	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-					
	59.019	27.238	13.742	0.001	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-					
	59.019	27.238	13.742	0.001	100.000
	SUM OF ROOTS	120.397		WARP FACTOR =	1.000

SET TWO

DATAPOINT	COORDINATES				
	1	2	3	4	5
1	−3.578	−2.497	−1.625	−0.016	−0.006
2	1.336	3.814	−1.466	−0.002	0.002
3	−0.869	−0.847	0.088	0.038	−0.001
4	−3.507	1.150	2.336	−0.011	−0.006
5	6.619	−1.621	0.666	−0.009	0.010
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--					
	71.449	25.450	10.698	0.002	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-					
	66.403	23.652	9.943	0.002	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-					
	66.403	23.652	9.943	0.002	100.000
	SUM OF ROOTS	107.599		WARP FACTOR =	1.000

TABLE THREE: COORDINATES OF FIVE POINTS AT TWO TIMES
TRANSLATED TO CENTER OF STABLE DATAPOINTS

GALILEO COORDINATES OF 5 VARIABLES IN A METRIC MULTIDIMENSIONAL SPACE FOR DATA SET 1

DATAPOINT	SOLUTION TRANSLATED TO STABLE CONCEPTS CENTROID				
	1	2	3	4	5
1	-3.693	-1.951	2.621	0.006	0.005
2	0.834	3.936	-0.355	0.017	-0.001
3	0.947	-3.721	-1.888	0.027	-0.001
4	-3.666	-0.518	-2.570	-0.013	0.005
5	6.524	-1.467	0.304	-0.010	-0.009
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--					
	71.236	35.564	17.258	0.001	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-					
	57.421	28.667	13.911	0.001	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-					
	57.421	28.667	13.911	0.001	100.000
	SUM OF ROOTS	124.059		WARP FACTOR =	1.000

GALILEO COORDINATES OF 5 VARIABLES IN A METRIC MULTIDIMENSIONAL SPACE FOR DATA SET 2

DATAPOINT	SOLUTION TRANSLATED TO STABLE CONCEPTS CENTROID				
	1	2	3	4	5
1	-3.796	-2.708	-1.603	-0.006	-0.006
2	1.118	3.603	-1.443	0.007	0.002
3	-1.086	-1.058	0.110	0.047	-0.002
4	-3.724	0.939	2.358	-0.002	-0.006
5	6.401	-1.833	0.688	0.001	0.010
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--					
	71.685	25.674	10.701	0.002	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-					
	66.337	23.758	9.902	0.002	0.000
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-					
	66.337	23.758	9.902	0.002	100.000
	SUM OF ROOTS	108.062		WARP FACTOR =	1.000

TABLE FOUR: ROTATED COORDINATES

THE ROTATED COORDINATES OF SPACE NUMBER 1

DATAPOINT	1	2	3	4	5
1	−3.693	−1.951	2.621	0.006	0.005
2	0.834	3.936	−0.355	0.017	−0.001
3	0.947	−3.721	−1.888	0.027	−0.001
4	−3.666	−0.518	−2.570	−0.013	0.005
5	6.524	−1.467	0.304	−0.010	−0.009

THE ROTATED COORDINATES OF SPACE NUMBER 2

DATAPOINT	1	2	3	4	5
1	−3.708	−1.968	2.586	−0.007	−0.006
2	0.839	3.938	−0.319	0.019	0.002
3	−1.017	−1.071	0.360	−0.057	−0.002
4	−3.652	−0.495	−2.594	−0.038	−0.006
5	6.522	−1.474	0.327	0.026	0.010

DISTANCES MOVED IN THE INTERVAL BETWEEN TIME 1 AND TIME 2

Concept 1 moved 0.043 units
Concept 2 moved 0.036 units
Concept 3 moved 3.993 units
Concept 4 moved 0.042 units
Concept 5 moved 0.039 units

TABLE FIVE: MAGNITUDES, SCALAR PRODUCTS, CORRELATIONS AND ANGLES BETWEEN POSITION VECTORS OF FIVE DATAPOINTS ACROSS TWO DATASETS

CONCEPT	T 1 MAGNITUDE	T 2 MAGNITUDE	SCALAR PRODUCT	CORRELATION	ANGLE
1	4.93	4.93	24.31	0.999963	0.5
2	4.04	4.04	16.31	0.999961	0.5
3	4.28	1.52	2.34	0.359398	68.9
4	4.51	4.51	20.31	0.999957	0.5
5	6.69	6.69	44.81	0.999983	0.3

TABLE SIX: DISTANCES AMONG FIVE POINTS IN A MULTIDIMENSIONAL RIEMANN SPACE

SET ONE				
DATAPOINT	DISTANCES			
	1	2	3	4
1	.000			
2	48.497	.000		
3	24.228	52.507	.000	
4	43.232	36.701	45.957	.000
5	68.330	45,771	85.241	65.100

SET TWO				
DATAPOINT	DISTANCES			
1	.000			
2	48.497	.000		
3	60.614	37.094	.000	
4	43.232	36.701	42.462	.000
5	68.330	45.771	27.074	65.100

TABLE SEVEN: COORDINATES OF 5 POINTS IN A RIEMANN SPACE

GALILEO COORDINATES OF 5 VARIABLES IN A METRIC MULTIDIMENSIONAL SPACE

DATAPOINT	1	2	3	4	5
		SET ONE		NORMAL SOLUTION	
1	− 19.533	− 17.153	− 6.936	− .023	.398
2	13.124	10.289	16.139	.014	.267
3	− 35.991	− 5.716	6.701	− .008	− .412
4	6.047	23.586	− 12.193	.032	− .031
5	48.447	− 11.005	− 3.712	− .015	− .222
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--	4232.725	1110.180	515.936	.002	− .450
NUMBER OF ITERATIONS TO DERIVE THE ROOT--	5	6	4	4	4
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-	72.251	18.950	8.807	.000	.008
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-	72.245	18.949	8.806	.446	100.437
	SUM OF ROOTS	5858.393			

GALILEO COORDINATES OF 5 VARIABLES IN A METRIC MULTIDIMENSIONAL SPACE

DATAPOINT	1	2	3	4	5
		SET TWO		NORMAL SOLUTION	
1	− 32.720	21.527	− 3.572	−.025	− .493
2	− .891	− 5.725	20.844	−.000	− .554
3	19.701	− 8.274	− 9.944	.016	− 2.121
4	− 21.091	− 20.070	− 6.312	−.016	1.528
5	35.002	12.542	− 1.015	.027	1.640
EIGENVALUES (ROOTS) OF EIGENVECTOR MATRIX--	3129.493	1124.776	586.993	.002	10.071
NUMBER OF ITERATIONS TO DERIVE THE ROOT--	5	5	4	4	4
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS-	64.777	23.282	12.150	.000	.208
PERCENTAGE OF VARIANCE ACCOUNTED FOR BY INDIVIDUAL FACTORS IN THEIR OWN SPACES-	64.642	23.233	12.125	.019	100.019
	SUM OF ROOTS	4831.192			

TABLE EIGHT: ROTATED COORDINATES OF 5 POINTS IN A RIEMANN SPACE

THE ROTATED COORDINATES OF SPACE NUMBER 1

DATAPOINT	1	2	3	4	5
1	− 28.530	− 18.582	− 5.261	− .025	.295
2	4.126	8.860	17.815	.012	.164
3	− 44.988	− 7.145	8.376	− .010	− .515
4	− 15.045	22.157	− 10.517	.030	− .134
5	39.449	− 12.434	− 2.037	− .017	− .325

THE ROTATED COORDINATES OF SPACE NUMBER 2

DATAPOINT	1	2	3	4	5
1	− 28.444	− 18.857	− 4.824	− .050	1.022
2	4.260	8.910	17.790	− .079	1.081
3	25.003	8.589	− 13.000	− .204	2.643
4	− 15.360	22.019	− 10.396	.082	− .995
5	39.544	− 12.072	− 2.570	.046	− 1.109

TABLE NINE: DISTANCES BETWEEN CORRESPONDING DATAPOINTS, MAGNITUDES, SCALAR PRODUCTS, CORRELATIONS AND ANGLES BETWEEN POSITION VECTORS OF 5 DATAPOINTS IN A RIEMANN SPACE

DISTANCES MOVED IN THE INTERVAL BETWEEN TIME 1 and TIME 2			
DATAPOINT	REAL	IMAGINARY	RIEMANN
1	.522	.728	− .507
2	.145	.921	− .910
3	74.855	3.164	74.788
4	.365	.862	− .781
5	.651	.786	− .441

CONCEPT	T 1 MAGNITUDE	T 2 MAGNITUDE	SCALAR PRODUCT	CORRELATION	ANGLE
1	34.45	34.45	1187.00	1.00	0.0
2	20.32	20.32	413.27	1.00	0.0
3	46.31	29.34	−1293.74	.95	162.2
4	28.77	28.77	828.18	1.00	0.0
5	41.41	41.41	1714.97	1.00	0.0

TABLE TEN: ROTATED COORDINATES OF 5 POINTS IN RIEMANN SPACE TO AN ORDINARY LEAST SQUARES SOLUTION

THE ROTATED COORDINATES OF SPACE NUMBER 1

DATAPOINT	1	2	3	4	5
1	−19.533	−17.153	−6.936	−.023	.398
2	13.124	10.289	16.139	0.14	.267
3	−35.991	−5.716	6.701	−.008	−.412
4	−6.047	23.586	−12.193	.032	−.031
5	48.447	−11.005	−3.712	−.015	−.222

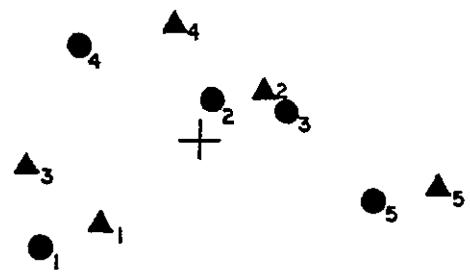
THE ROTATED COORDINATES OF SPACE NUMBER 2

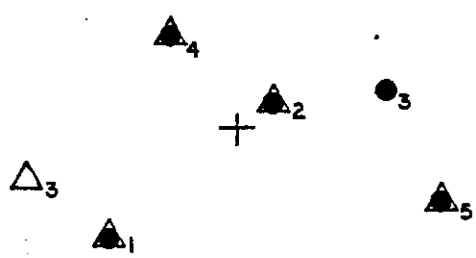
DATAPOINT	1	2	3	4	5
1	−32.140	−22.438	3.222	−.060	−.490
2	1.719	7.930	20.055	−.096	−.545
3	17.889	7.541	−13.363	−.383	−2.086
4	−22.429	18.843	−5.422	.281	1.502
5	34.961	−11.875	−4.491	.258	1.619

TABLE ELEVEN: DISTANCES BETWEEN CORRESPONDING DATAPOINTS, MAGNITUDES, SCALAR PRODUCTS, CORRELATIONS AND ANGLES AMONG THEIR POSITION VECTORS (ORDINARY LEAST SQUARES SOLUTION).

DISTANCES MOVED IN THE INTERVAL BETWEEN TIME 1 AND TIME 2			
DATAPOINT	REAL	IMAGINARY	RIEMANN
1	17.031	.888	17.007
2	12.287	.820	12.260
3	59.003	1.716	58.978
4	18.349	1.553	18.284
5	13.536	1.862	13.408

CONCEPT	T 1 MAGNITUDE	T 2 MAGNITUDE	SCALAR PRODUCT	CORRELATION	ANGLE
1	26.90	39.33	990.51	.936252	20.6
2	23.21	21.63	427.97	.852744	31.5
3	37.05	23.47	−777.36	−.893854	153.4
4	27.23	29.75	646.20	.797614	37.1
5	49.82	37.16	1841.46	.994737	5.9





FIGURES

Figure 1. Two datasets expressed on common (rotated) coordinates. Figure 1a gives the results of ordinary least squares rotation, while figure 1b gives the result of using only the 1st, 2nd, 4th and 5th datapoints as a reference frame. Triangles represent dataset 1.

FOOTNOTES

1. In typical treatments, Riemann surfaces are defined in general by the differential form given in (3), since a Riemann space is, in general, curvilinear. It is always possible, however, to project a curved Riemann surface into a linear Riemann space of higher dimensionality (although, particularly in the case of hyperbolic surfaces, this may result in the presence of cusps or edges, which may be given substantive interpretations by catastrophe theory). In the case considered here, we suffer no important loss of generality by referring always to the flat Riemann space, because metric multidimensional scaling operations such as those discussed here will always produce the larger flat space in preference to the smaller curved surfaces if the number of dimensions is left a free parameter. In our own practical experience, as with that of our colleagues, several hundred empirical cases have never resulted in a case in which the generalized form (1) fails to regenerate the original dissimilarities matrix to within trivial rounding error.
2. These data were analyzed by means of the Galileo™ version 5.2 computer program at the State University of New York at Albany.

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