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1.1. The Problem

While numerous Multidimensional Scaling Procedures have been developed over the past several years, all of them share a common assumption: that each of the objects or "stimuli" to be scaled may be represented as a point in space. While different workers have presented a variety of assumptions about the topological properties of the space, few have questioned the assumption about the point-like character of the stimuli themselves.

However common this assumption may be, it is neither necessary nor even compelling in many circumstances. In the classical example of scaling color terms, for example, it is clearly inappropriate to represent the range of the spectrum denoted by the English terms "yellow" or "blue" as points, since it is well known that these terms refer to a line--segment on the color spectrum. Similarly, in a political scaling exercise, terms such as "the Democratic Party" might well be thought of as spanning some non--zero distance along, for example, a liberal-conservative continuum. For the case in which stimuli of this kind are displayed along multiple dimensions, the sets of line-segments that correspond to the coordinates on these dimensions will generally intersect to form regions that distinguish stimuli from one another. Figure 1 illustrates one such possibility.

Figure 1 about here

Figure 1 represents an hypothetical example in which several autos have been rated for their speed and beauty. In view of the fact that brands of autos generally produce a range of models that are likely to exhibit differing values on these two dimensions, it seems reasonable to use regions to depict brands such that each region corresponds to a brand name that refers to several models. While there is an inherent ambiguity associated with the respective brand-names it is not so great that the brands are indistinguishable from one another. (For simplicity, we assume that speed and beauty are orthogonal attributes, although this assumption is not necessary to the system under consideration.)

1.1.1. Non-euclidian models

Classical Multidimensional Scaling usually assumes that the initial dissimilarities, s(i,j) meet the "triangle inequality" constraints, that is, s(i,j)+ s(i,m)) s(j,m) for all values of i,j and m. For k stimuli, data of this form compose a Euclidean space of at most k-1 dimensions.

When data fail to meet these constraints, as data from the psychological and cultural sciences often do, they may still be represented in the same general form, but in this case each point in the space has k-1 coordinate values, p of which will be real and r-p of which will be imaginary. (When the dissimilarities matrix is also asymmetric, each of the k-1 coordinates will be complex, but this case is not considered here.)

Several workers have recently considered Multidimensional Scaling in general Riemann spaces (Piesko, 1970; Woelfel and Barnett, 1982). Data fail the triangle inequalities rule when two points, i and j, are relatively close to the same third point, m, but are not close to one another. In Riemann scaling, this discrepancy is modeled by a warped space in which the distance between i and j is not measured on a straight line, but along a geodesic or "straightest line" on the curve of the space between i and j. Riemann spaces have been studied carefully by mathemeticians and physicists, and several useful treatments may be found in the psycnometric literature (cf. Woelfel and Barnett, 1982). Further investigations along these lines show great promise.

However feasible such Riemannian systems may be, applied scientists usually find visualizations of the Riemannian space difficult, and this may seriously hinder substantive developments.

The major thesis of this paper is that failure to meet the triangle inequality can often be explained by representing the "stimuli" as k hyperspheres of given radii, rather than as k points, whose centers have r coordinates in the Euclidean space R.

Within this framework, a stimulus is "large" to the extent to which it can be close to several other stimuli that are not close to one another. In typical applied uses of MDS, stimuli often correspond to beliefs, ideas and the like, and stimuli that are large in the sense offered here would correspond to stimuli of general meaning, covering a large volume of psychological or cultural space, while small stimuli would represent ideas of relatively precise meaning.

In communication network studies, where data often consist of frequencies or inverse frequencies of communication among members of a group or organization, some individuals may communicate frequently with two other individuals who communicate infrequently with one another. Such a person might therefore be close to two individuals who are not close to each other. In the model elaborated here, such a person would be described as "large."

Data input into multidimensional scaling algorithms frequently consist of distance or dissimilarites matrices which have been averaged over the responses of many subjects. When such distances have been averaged across several judges it would be reasonable to argue that "large" concepts may be regarded as covering an imprecisely defined joint conception -- that is, a concept is ambiguous in the sense that different judges may perceive it quite differently from one another. The same judge may also perceive a given set of objects as different from one another on different occasions.

The regions corresponding to stimuli might also be identified with "fuzzy sets" as that term has recently been amplified in certain mathematical literature. In general all the cases we have referenced refer either to ambiguity in the meaning of concepts or to dynamic change among concepts across time or across respondents. Many of the applications of the fuzzy set concept similarily deal with either ambiguity or dynamic change.

1.2. Methods

In its most general form, the theory underlying our approach places no restrictions on the sizes or shapes of the stimuli. We present here a solution to a more restricted model. Specifically, the solution offered is appropriate only for dissimilarites or distance matrices which are symmetric, and whose values are all positive. We assume that the stimuli may be represented as hyperspheres, not hyperboxes, or other complex shapes, so that the volume of each region can be characterized by a single parameter. The extent to which these simplifying assumptions are adequate to different scaling tasks must be established by experiments on a case by case basis, but our initial speculations and some examples suggest that data in which the interstimulus distances are large relative to the sizes of the stimuli should meet these assumptions to a reasonable approximation.

The model presented here assumes that different stimuli may be represented as circles, spheres, or hyperspheres of varying radii, and that the observed distances between them, $s(i,j)$, are shortest surface-to-surface distances. As Figure 2 illustrates. these distances would be smaller than the center to center distances $s'(i,j)$ by the sum of these radii:

$$s'(i,j) = s(i,j) + r(i) + r(j).$$

Thus if

$$s*(i,j) = r(i) + r(j),$$

then,

$$s'(i,j) = s(i,j) + s*(i,j).$$

One may assume that any failures of the triangle inequality relations among a set of stimuli may be attributed to variations in areas or volumes. Thus, we might assume that estimates of the dissimilarites among a set of stimuli derive from surface--to--surface distances among hyperspheres representing stimuli rather than from center--to--center distances, and that this in turn will generally cause the resulting dissimilarities to fail the triangle inequalities relation. As suggested above, in this case some of the eigenvectors of the double--centered scalar products matrix derived from these dissimilarities will be imaginary (Torgerson, 1958, chap. 11).

In the present paper, we discuss two such functions. For the first approach we partition the space into its real and imaginary components. The distances or dissimilarities among the stimuli within the imaginary part of the space are considered in this context to be a simple additive function of the radii of the stimuli, such that

$$s*(i,j) = r(i) + r(j), \quad i=1,\ldots,k;\ j=i+1,\ldots,k.$$

where $s*(i,j)$ = the absolute distances among the stimuli in the imaginary part of the space, and

$r(i)$ & $r(j)$ = the radii of the ith and jth stimuli

then using standard techniques (cf. Carroll and Pruzansky, 1980 or Zinnes & MacKay, 1983) it can be shown that the radii may be estimated as

$$r(i) = (D(i) - (1/[k-1])T)/[k-2], \quad i = 1,..,k.$$

where $D(i)$ denotes the sum of the $s*(i,j)$ for the ith stimulus across j, and T is the average of the $D(i)$ across i.

A second approach considered in this paper is equally simple: we may assume that the distance each stimulus projects into the imaginary part of the space is a function of its radius. Estimates of the radii under this assumption consist simply of the lengths of the position vectors of the stimuli in the imaginary part of the space, or

$$r(i) = sqrt(r(i,l)**2), \quad l=p+1,...,r.$$

1.3. Examples

Data for testing consisted of arbitrary arrays of circles of varying radii on the plane. The number of circles in each array varied from 4 to 20, with one space of 4 circles, one of 5 circles, 20 of 6 circles, one of 7 circles, one of 10 circles, one of 14 circles and 7 of 20 circles.

Within each of these examples, distances were measured between the respective perimeters of each pair of circles to the nearest millimeter, and each of the resulting complete pair-comparison matrices were entered into the Galileo (Version 5.2) scaling program (Woelfel & Fink, 1980). The Galileo scaling program computes a double-centered scalar products matrix (B*) following Torgerson (1958, Chapt. 11), and extracts all non-zero eigenvectors, both real and imaginary, by an iterative procedure based on Van de Geer (1972). Each of these eigenvectors is normalized to the corresponding diagonal element of the matrix B*, so that the distances within the resulting coordinate system correspond exactly to the original measured distances. The coordinates from the Galileo outputs serve as input to the equations described above. True radii were then regressed on the estimates provided by the equations described above.

Correlations between the estimates and the true radii are presented in Table 1. As these figures show, both methods work reasonably well in most cases, and very well in some cases, although four cases exist in which correlations between the estimated radii and the true radii are essentially zero. At this stage of our investigation we have uncovered no systematic reason why the solutions should fail in these cases.

Table 1: Correlations of Two Solutions with True Radii

Dataset	Method I	Method II
20 Circles (1)	.95	.91
20 Circles (2)	.72	.71
20 Circles (3)	.64	.64
20 Circles (4)	.74	.76
20 Circles (5)	.90	.90
20 Circles (6)	.86	.81
20 Circles (7)	.89	.89
14 Circles	.85	.85
10 Circles	.71	.74
7 Circles	.91	.91
6 Circles (1)	.48	.54
6 Circles (2)	.69	.70
6 Circles (3)	.82	.74
6 Circles (4)	.82	.79
6 Circles (5)	.83	.83
6 Circles (6)	.53	.56
6 Circles (7)	.77	.81
6 Circles (8)	.14	.07
6 Circles (9)	.76	.82
6 Circles (10)	.03	.00
6 Circles (11)	.64	.65
6 Circles (12)	.51	.54
6 Circles (13)	.59	.74
6 Circles (14)	.50	.77
6 Circles (15)	.04	.00
6 Circles (16)	.05	.12
6 Circles (17)	.52	.57
6 Circles (18)	.62	.71
6 Circles (19)	.51	.57
6 Circles (20)	.90	.83
5 Circles	.82	.75
4 Circles	.99	.96

1.4. Discussion

The present analysis should be considered only a preliminary discussion of the problem. The examples considered are very limited, both in number and in scope. All examples except one consist of circles in the plane, and data spanning more than two dimensions have not been considered. The single exception is the case referenced as "14 Circles" in Table 1, and in this case all the points are arrayed in a straight line. Radii recovered in this case might well be considered interval estimates in the unidimensional scaling case, which may be of special interest in itself.

Since such a limited array of potential solutions have been considered, we cannot rule out the possibility that a superior solution exists, but the level of recovery of radii in most cases is probably good enough in the present case to be of some substantive use to applied investigators.

Figure 1. Speed and Beauty of Selected Autos

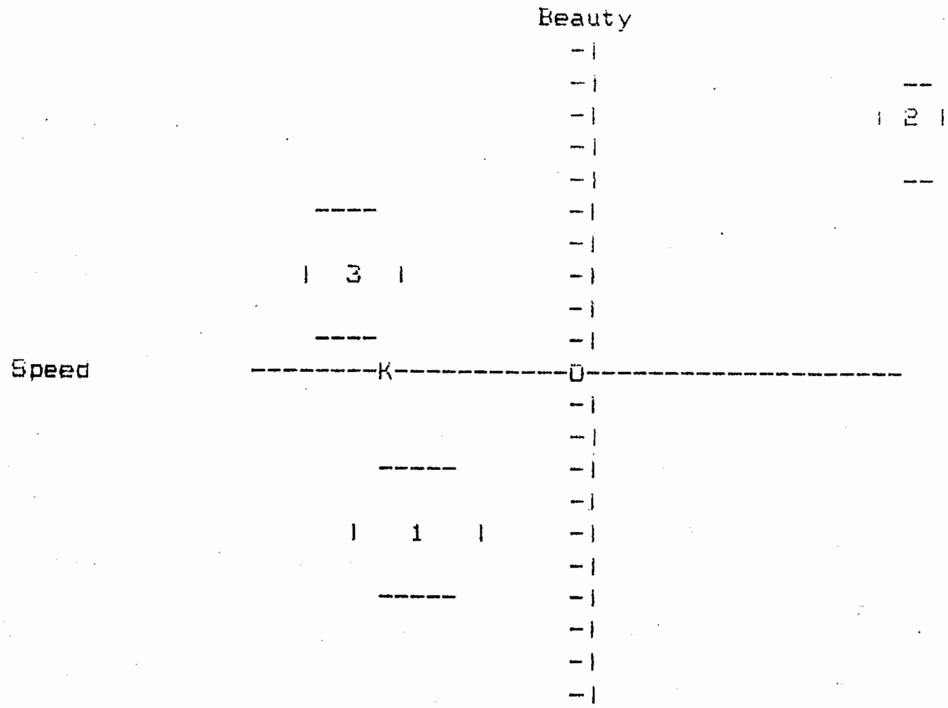

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Figure 2. Distances Among Stimuli of Varying Radii

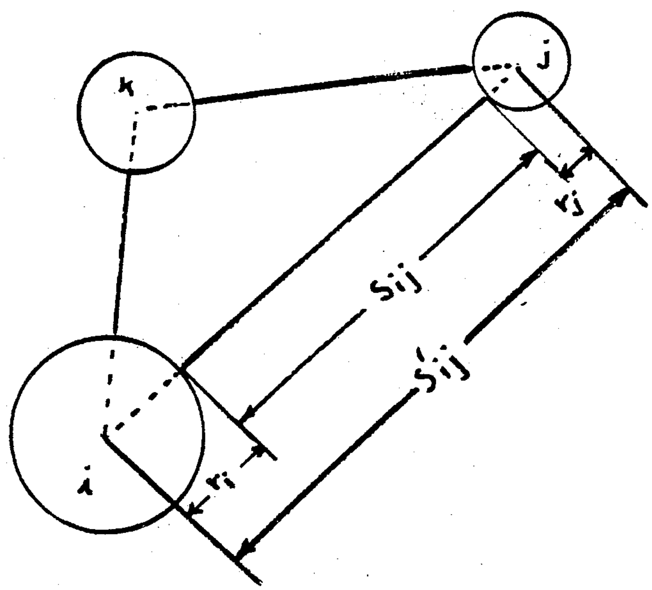

Figure 1. Speed and Beauty of Three Autos